Assignment 2

Hand in no. 1 and 2 by September 19, 2023.

1. Let $C^\infty_{2\pi}$ be the class of all smooth 2π-periodic, complex-valued functions and $\mathcal{C}^\infty$ the class of all complex bisequences satisfying $c_n = \circ(n^{-k})$ as $n \to \pm\infty$ for every k. Show that the Fourier transform $f \mapsto \hat{f}$ is bijective from $C^\infty_{2\pi}$ to $\mathcal{C}^\infty$. Hint: You need to apply those theorems on uniform convergence in MATH2060.

2. Propose a definition for $\sqrt{d/dx}$. This operator should be a linear map which maps $C^\infty_{2\pi}$ to itself satisfying
$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f = \frac{d}{dx}f,$$
for all smooth, 2π-periodic f.

3. Let f be a continuous, 2π-periodic function and its primitive function be given by
$$F(x) = \int_0^x f(x)dx.$$
Show that F is 2π-periodic if and only if f has zero mean. In this case,
$$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.$$

4. Let $\mathcal{C}'$ be the subspace of $\mathcal{C}$ consisting of all bisequences $\{c_n\}$ satisfying $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$.
   (a) For $f \in R[-\pi, \pi]$, show that
   $$2\pi \sum_{-\infty}^{\infty} |c_n|^2 \leq \int_{-\pi}^{\pi} |f|^2 \ .$$
   (b) Deduce from (a) that the Fourier transform $f \mapsto \hat{f}(n)$ maps $R_{2\pi}$ into $\mathcal{C}'$.
   (c) Explain why the trigonometric series
   $$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha} \ , \quad \alpha \in (0, 1/2] \ ,$$
   is not the Fourier series of any function in $R_{2\pi}$.